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Hamiltonian cycles and paths through matchings

Denise Amar [1]

LaBRI Domaine Universitaire
351, cours de la Libération, 33405 Talence Cedex France

Evelyne Flandrin [2]

Laboratoire de Recherche en Informatique
Bât 490 Université Paris-Sud, 91405 Orsay Cedex France

Grzegorz Gancarzewicz [3,4]

AGH University of Science and Technology
Wydział Matematyki Stosowanej
Al. Mickiewicza 30, 30-059 Kraków, Poland

Abstract

We give sufficient degree-sum conditions for a graph to contain every matching in a hamiltonian cycle or a hamiltonian path. Moreover we prove that some results are almost best possible.

Keywords: Cycle, hamiltonian cycle, matching.

[1] Email: `amar@labri.u-bordeaux.fr`
[2] Email: `fe@lri.fr`
[3] Email: `gancarz@uci.agh.edu.pl`
[4] This work was carried out in part while GG was visiting, LRI UPS Orsay, France.

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1 Introduction

In 1960 O. Ore [6] proved the following:

Theorem 1.1 *Let G be a graph on $n \geqslant 3$ vertices. If for all nonadjacent vertices $x, y \in \mathrm{V}(G)$ we have: $\mathrm{d}(x) + \mathrm{d}(y) \geqslant n$, then G is hamiltonian.*

Later many Ore type theorems dealing with degree-sum conditions where proved. In particular J.A. Bondy [2] proved:

Theorem 1.2 *Let G be a 2-connected graph on $n \geqslant 3$ vertices. If for any independent vertices $x, y, z \in \mathrm{V}(G)$ we have: $\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant \frac{3n-2}{2}$, then G is hamiltonian.*

Let G be a graph and let $k \geqslant 1$. We shall call a set of k independent edges a *k-matching* or simply a *matching.*

About cycles through matchings in general graphs K.A. Berman [1] proved the following result conjectured by R. Häggkvist [4].

Theorem 1.3 *Let G be graph of order n. If for any x, $y \in V(G)$, $xy \notin E$ we have $\mathrm{d}(x) + \mathrm{d}(y) \geqslant n + 1$, then every matching lies in a cycle.*

Theorem 1.3 has been improved by B. Jackson and N.C. Wormald in [5]. R. Häggkvist [4] gave also a sufficient condition for a general graph to contain any matching in a hamiltonian cycle. We give this theorem below in a slightly improved version obtained in [7] by A.P. Wojda.

We denote the *join* of graphs by $*$.

Let $\mathcal{G}_n$ be the family of graphs $G = \overline{K}_{\frac{n+2}{3}} * H$, where H is any graph of order $\frac{2n-3}{3}$ containing a perfect matching, if $\frac{n+2}{3}$ is an integer, and $\mathcal{G}_n = \emptyset$ otherwise.

Theorem 1.4 *Let G be a graph of order $n \geqslant 3$, such that for every pair of nonadjacent vertices x and y $d(x) + d(y) \geqslant \frac{4n-4}{3}$. Then every matching of G lies in a hamiltonian cycle, unless $G \in \mathcal{G}_n$.*

2 Results

We have proved a similar result to Theorem 1.4, for containing every matching in a hamiltonian path.

Theorem 2.1 ([3]) *Let $s \geqslant 1$ and let G be a graph of order $n \geqslant 4s+6$, such that for every pair of nonadjacent vertices x and $y \in \mathrm{V}(G)$ we have: $\mathrm{d}(x)+\mathrm{d}(y) \geqslant \frac{4n-4s-3}{3}$, then every matching of G lies on a cycle of length at least $n-s$ and hence in a path of length at least $n-s+1$.*

Suppose that $s \geqslant 1$ is such that $n \geqslant 4s+6$ and $\frac{n+2s}{3} \geqslant 2$ is an integer. Let H be a graph of order $\frac{2n-2s}{3}$ containing a perfect matching.

Consider now the graph $G' = (\frac{n+2s}{3} - 1)\, K_1 \star H$.

We shall define a graph G'' as a graph obtained from G' by adding an external vertex x adjacent only to $\frac{2n-2s}{3} - 1$ vertices from H i.e. we take $\mathrm{V}(G'') = \mathrm{V}(G') \cup \{x\}$, next we choose an arbitrary vertex $h_0 \in \mathrm{V}(H)$ and we put $\mathrm{E}(G'') = \mathrm{E}(G') \cup \{xh : h \in \mathrm{V}(H) \setminus \{h_0\}\}$.

The graph G'' satisfies the assumptions of Theorem 2.1 but violates those of Theorem 1.4. So Theorem 1.4 and Theorem 2.1 are independent.

Note that the most interesting case of Theorem 2.1 is the case $s = 1$, because we have a hamiltonian path containing M i.e.

Corollary 2.2 *Let G be a graph of order $n \geqslant 10$, such that for every pair of nonadjacent vertices x and $y \in \mathrm{V}(G)$ we have: $\mathrm{d}(x)+\mathrm{d}(y) \geqslant \frac{4n-7}{3}$, then every matching of G lies in a cycle of G of length at least $n-1$ and hence in a hamiltonian path.*

It is easy to check that even this case of Theorem 2.1 can not be obtained as a corollary of Theorem 1.4 by adding in the graph G an external vertex x adjacent to all vertices and removing an edge from a hamiltonian cycle in $G \cup \{x\}$. In this case $G \cup \{x\}$ does not satisfy the assumptions of Theorem 1.4.

Next we have tried to find a degree sum condition for three independent vertices under which every matching from a graph G is contained in a hamiltonian cycle.

Let G be a graph. We define $\alpha(G)$, *the stability number of* G, as the cardinal of a maximum independent set of vertices of G. For a $H \subset \mathrm{E}(G)$, F is an H-edge cut-set of G if and only if $F \subset H$ and $G \setminus F$ is not connected. F is said to be a minimal H-edge cut-set of G if and only if F is an H-edge cut-set of G which has no proper subset being an H-edge cut-set.

We have first proved the following theorem:

Theorem 2.3 *Let G be a 3-connected graph of order $n \geqslant 3$ and let M be a matching of G. If $\alpha(G) = 2$, then there is a hamiltonian cycle of G containing M or G has a minimal odd M-edge cut-set.*

Later we have obtained the following result:

Theorem 2.4 *Let G be a 3-connected graph of order $n \geqslant 3$, let $0 \leqslant k \leqslant \frac{n}{3}$ and let M be a k-matching of G. If for any independent vertices $x, y, z \in \mathrm{V}(G)$ $\mathrm{d}(x)+\mathrm{d}(y)+\mathrm{d}(z) \geqslant 2n$, then there is a hamiltonian cycle of G containing M or G has a minimal odd M-edge cut-set.*

We can show a similar theorem without the assumption that $k \leqslant \frac{n}{3}$, but in this case $\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant \frac{9n}{4}$.

Note that the bound $2n$ in Theorem 2.4 is almost best possible. Let $p \geqslant 2$ and consider a complete graph K_{2p} with a perfect p-matching. We define the graph $G = (p+1)K_1 * K_{2p}$, ($*$ denotes the join of graphs). In this graph $n = 3p+1$ and $p \approx \frac{n}{3}$. For any independent $x, y, z \in \mathrm{V}(G)$ $\mathrm{d}(x)+\mathrm{d}(y)+\mathrm{d}(z) \geqslant 2n-2$ and there is no hamiltonian cycle containing the p-matching from K_{2p}. So the bound $2n$ is almost best possible.

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