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Note

# On the maximum number of cycles in a Hamiltonian graph

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**Abstract**

Let $M(k)$ denote the maximum number of cycles in a Hamiltonian graph of order $n$ and size $n+k$. We prove that $M(k) \geqslant 2^k + (5/2)k^2 - (21/2)k + 14$ and $M(k) \leqslant 2^{k+1} - 1 - k\left(\frac{\sqrt{k}-2}{\log_2(k)+2} - \frac{1}{4}\log_2(k)\right)$ for $k \geqslant 4$. Furthermore, we determine $M(k)$ and the structure of the extremal graphs for $5 \leqslant k \leqslant 10$ exactly. Our results give partial answers to a problem raised by Shi [The number of cycles in a hamilton graph, Discrete Math. 133 (1994) 249–257].


*Keywords:* Cycle; Hamiltonian graph; Number of cycles

## 1. Introduction

We consider finite, simple graphs and use standard terminology (cf. e.g. [1]). Motivated by questions posed by Yap and Teo in [5], Shi [3] studied the maximum number of cycles $M(k)$ in a Hamiltonian graph $G=(V(G), E(G))$ of order $n=|V(G)|$ and size $n+k=|E(G)|$. Let $\Gamma_k$ denote the set of all such graphs. Clearly, if $f(G)$ denotes the number of cycles of a graph $G$, then $M(k) = \max\{f(G) \mid G \in \Gamma_k\}$.

Throughout the paper we will tacitly assume that for each graph $G$ in $\Gamma_k$ one Hamiltonian cycle $C_G : u_1u_2 \ldots u_n$ has been fixed and we call the additional edges in $E(G)\setminus E(C_G)$ *chords*. Two chords $u_1v_1$ and $u_2v_2$ of some $G \in \Gamma_k$ are called *skew*, if their endvertices

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appear in the cyclic order $u_1$, $u_2$, $v_1$, $v_2$ on $C_G$. Two chords are *parallel*, if they are not skew.

Shi's main results in [3] concerning $M(k)$ are that for $k \geqslant 1$

$$2^k + k(k-1) + 1 \leqslant M(k) \leqslant 2^{k+1} - 1 \tag{1}$$

and that for $1 \leqslant k \leqslant 4$ equality holds in the left inequality of (1).

We will first prove some properties of the extremal graphs, i.e. those graphs $G$ in $\Gamma_k$ for which $f(G) = M(k)$. As our main results we improve (1) and determine $M(k)$ and the structure of the extremal graphs exactly for $5 \leqslant k \leqslant 10$.

Our research was initially motivated by the hope of finding some regularity in the structure of the extremal graphs which would have allowed to determine $M(k)$ exactly for all (or several) $k$. Unfortunately, we were unable to do so and have to leave it as a challenging open problem. The interested reader may find similar results in [2] or [4] and the references mentioned therein.

## 2. Results on $M(k)$ for general $k$

We need some more notation and observations already used by Shi. Let $G \in \Gamma_k$ and let $S \subseteq E(G) \backslash E(C_G)$. The number of cycles $C$ of $G$ such that $S = E(C) \backslash E(C_G)$ is denoted by $g_G(S)$. Therefore, $f(C) = \sum_{S \subseteq E(G) \backslash E(C_G)} g(S)$. Shi observed that $g_G(S) \leqslant 2$ for all $S \subseteq E(G) \backslash E(C_G)$ (Theorem 3.1 in [3]).

The next lemma describes some properties of extremal graphs.

**Lemma 1.** (i) *If $G \in \Gamma_k$ and $f(G) = M(k)$, then no two chords are incident with the same vertex, i.e. the maximum degree $\Delta(G)$ of $G$ is at most* 3.

(ii) *If $G \in \Gamma_k$ with $f(G) = M(k)$, then there is no set of chords $S \subseteq E(G) \backslash E(C_G)$ such that $\emptyset \neq S \neq E(G) \backslash E(C_G)$ and each chord in $S$ is parallel to each chord not in $S$.*

(iii) *For $k \geqslant 2$ there is a* 3*-regular graph $G \in \Gamma_k$ with $f(G) = M(k)$.*

(iv) *For $k \geqslant 2$ every graph $H \in \Gamma_k$ with $f(H) = M(k)$ arises from a* 3*-regular graph $G \in \Gamma_k$ with $f(G) = M(k)$ by replacing the edges of $C_G$ by non-trivial paths.*

**Proof.** (i) Assume to the contrary that there are chords $u_1v, u_2v \in E(G) \backslash E(C_G)$. Let $N_G(v) \cap V(C_G) = \{v^-, v^+\}$ and let the graph $G' \in \Gamma_k$ arise from $G$ by replacing $v$ by a path $v_1v_2$ on two vertices such that $N_{G'}(v_1) \cap V(C_G) = \{v^-\}$, $N_{G'}(v_2) \cap V(C_G) = \{v^+\}$ and replacing $u_1v, u_2v$ by $u_1v_1, u_2v_2$ such that $u_1v_2$ and $u_2v_2$ are skew (this can be ensured by appropriately renaming of $u_1, u_2$).

Clearly, to every cycle of $G$ corresponds a unique cycle of $G'$. Furthermore, we have $g_G(\{u_1v, u_2v\}) = 1 < 2 = g_{G'}(\{u_1v_2, u_2v_2\})$ which implies the contradiction $f(G') > f(G)$.

(ii) Assume to the contrary that such a set $S = \{u_1v_1, u_2v_2, \ldots, u_lv_l\}$ exists.

We may assume that $uv \in E(G) \backslash (E(C_G) \cup S)$ is such that $u$ and $u_1$ partition $C_G$ into two paths $P_1$ and $P_2$ such that all interior vertices of $P_1$ have degree 2. Let $G'$ arise from $G$ by replacing $uv$ and $u_1v_1$ by $uv_1$ and $u_1v$.

Clearly, to every cycle of $G$ that does not contain both of $uv$ and $u_1v_1$ corresponds a unique cycle of $G'$. By the properties of $S$, every cycle of $G$ that contains $uv$ and $u_1v_1$ must go along $P_1$. This implies that also to every cycle of $G$ that contains both of $uv$ and $u_1v_1$ corresponds a unique cycle of $G'$. Finally, $g_G(\{uv, u_1v_1\}) = 1$ and $g_{G'}(\{uv_1, u_1v\}) = 2$ which implies the contradiction $f(G') > f(G)$.

(iii) and (iv) are immediate consequences of (i) and (ii). □

Before we proceed to the main result of this section we give some auxiliary results. The first lemma deals with the number of cycles in two special graphs and the second lemma is a straightforward consequence of Ramsey theory.

**Lemma 2.** (i) [cf. Theorem 4.3 in [3]] *If $G \in \Gamma_k$ is such that $C_G : u_1u_2 \dots u_kv_1v_2 \dots v_k$ and $E(G)\setminus E(C_G) = \{u_iv_i \mid 1 \leqslant i \leqslant k\}$, then $f(G) = 1 + 2^k + 2\binom{k}{2}$.*

(ii) *If $G \in \Gamma_k$ is such that $C_G : u_1u_2 \dots u_kwv_kv_{k-1} \dots v_1w'$ and $E(G)\setminus E(C_G) = \{u_iv_i \mid 1 \leqslant i \leqslant k\}$, then $f(G) = 1 + 2k + \binom{k}{2}$.*

**Proof.** (i) Was proved by Shi [3] and (ii) is easy and left to the reader. □

**Lemma 3.** *Let $G = (V(G), E(G))$ be a graph of order $n$ and let $r = \lfloor \log_2(\sqrt{n}) + 1 \rfloor$.*

*There are $\lfloor n/2r \rfloor + 1$ disjoint subsets of $V(G)$ of size $r$ such that each of the sets induces either a clique or an independent set.*

**Proof.** Let $l = \lfloor n/2r \rfloor + 1$. Note that, by the choice of $r$ and $l$, $n/2 \geqslant 2^{2r-3}$ and $n - (l-1)r \geqslant n/2$.

It is well-known from Ramsey theory (cf. e.g. [1]) that every graph of order at least $2^{2r-3}$ has either a clique of size $r$ or an independent set of size $r$. Repeatedly deleting such sets from the graph yields the desired result. □

**Theorem 1.** *For $k \geqslant 5$*

$$M(k) \geqslant 2^k + \frac{5}{2}k^2 - \frac{21}{2}k + 14 \tag{2}$$

*and*

$$M(k) \leqslant 2^{k+1} - 1 - k\left(\frac{\sqrt{k} - 2}{\log_2(k) + 2} - \frac{1}{4}\log_2(k)\right). \tag{3}$$

**Proof.** In order to prove (2) we determine $f(H)$ for the graph $H \in \Gamma_k$ with $C_H : u_1u_2u_3u_4u_5u_6 \dots u_kv_2v_1v_4v_3v_5v_6 \dots v_k$ and $E(H)\setminus E(C_H) = \{e_i = u_iv_i \mid 1 \leqslant i \leqslant k\}$. We make repeated use of the following two claims, the first of which is obvious and the second of which has a simple proof that we leave to the reader.

**Claim 1.** *Contracting edges that are incident to a vertex of degree 2 or subdividing edges does not change the number of cycles of a graph.*

**Claim 2.** *Let $G \in \Gamma_k$ and let $S \subseteq E(G)\setminus E(C_G)$ contain two edges $a_1b_1$ and $a_2b_2$ that are skew such that $a_1a_2, b_1b_2 \in E(C_G)$. Let $G' = (V(G'), E(G'))$ be such that $V(G') =$*

*$V(G) \cup \{a', a'', b', b''\}$ and*

$$E(G') = (E(G)\backslash\{a_1a_2, b_1b_2\}) \cup \{a_1a', a'a'', a''a_2, b_1b', b'b'', b''b_2\} \cup \{a'b', a''b''\}.$$

(i) *If every cycle $C$ of $G$ with $S = E(C)\backslash E(C_G)$ satisfies $|E(C) \cap \{a_1a_2, b_1b_2\}| = 1$, then $g_G(S) = g_{G'}(S \cup \{a'b', a''b''\})$.*
(ii) *If every cycle $C$ of $G$ with $S = E(C)\backslash E(C_G)$ satisfies $|E(C) \cap \{a_1a_2, b_1b_2\}| \neq 1$, then $g_{G'}(S \cup \{a'b', a''b''\}) = 0$.*

We now consider the different subsets of $S \subseteq E(H)\backslash E(C_H)$.

Clearly, $g_H(\emptyset) = 1$.

Since $g_H(\{e_1, e_2, e_3, e_4, e_5\})=2$, Claims 1 and 2 imply $g_H(S)=2$ for all $S \subseteq E(H)\backslash E(C_H)$ with $|S \cap \{e_1, e_2, e_3, e_4\}| = 4$ and $|S|$ odd. There are $2^{k-5}$ such sets.

Since $g_H(\{e_1, e_2, e_3, e_4\})=g_H(\{e_1, e_2, e_3, e_4, e_5, e_6\})=1$, Claims 1 and 2 imply $g_H(S)=1$ for all $S \subseteq E(H)\backslash E(C_H)$ with $|S \cap \{e_1, e_2, e_3, e_4\}| = 4$ and $|S| \in \{4, 6\}$. There are $1 + \binom{k-4}{2}$ such sets.

Since $g_H(\{e_1, e_2, e_5\}) = 2$, Claims 1 and 2 imply $g_H(S) = 2$ for all $S \subseteq E(H)\backslash E(C_H)$ with $S \cap \{e_1, e_2, e_3, e_4\} \in \{\{e_1, e_2\}, \{e_3, e_4\}\}$ and $|S|$ is odd. There are $2 \cdot \left(2^{k-5} + 2 \cdot 2^{k-5}\right)$ such sets.

Since $g_H(\{e_1, e_2\}) = g_H(\{e_1, e_2, e_5, e_6\}) = 1$, Claims 1 and 2 imply $g_H(S) = 1$ for all $S \subseteq E(H)\backslash E(C_H)$ with $S \cap \{e_1, e_2, e_3, e_4\} \in \{\{e_1, e_2\}, \{e_3, e_4\}\}$ and $|S| \in \{2, 4\}$. There are $2 + 2 \cdot \left(\binom{k-4}{2} + 2 \cdot (k-4)\right)$ such sets.

Since $g_H(\{e_1\}) = g_H(\{e_1, e_3, e_5\}) = 2$, Claims 1 and 2 imply $g_H(S) = 2$ for all $S \subseteq E(H)\backslash E(C_H)$ with $|S \cap \{e_1, e_2\}| \neq 2$, $|S \cap \{e_3, e_4\}| \neq 2$ and $|S|$ is odd. There are $2^{k-5} + 4 \cdot 2^{k-5} + 4 \cdot 2^{k-5}$ such sets.

Since $g_H(\{e_1, e_3\}) = 2$, Claims 1 and 2 imply $g_H(S) = 2$ for all $S \subseteq E(H)\backslash E(C_H)$ with $|S \cap \{e_1, e_2\}| \neq 2$, $|S \cap \{e_3, e_4\}| \neq 2$ and $|S| = 2$. There are $\binom{k-4}{2} + 4 + 4(k-4)$ such sets.

Altogether, we obtain

$$\begin{aligned} f(H) &= \sum_{S \subseteq E(H)\backslash E(C_H)} g_H(S) \\ &\geqslant 1 + 2 \cdot 2^{k-5} + 1 + \binom{k-4}{2} + 2 \cdot 2 \cdot \left(2^{k-5} + 2 \cdot 2^{k-5}\right) \\ &\quad + 2 + 2 \cdot \left(\binom{k-4}{2} + 2 \cdot (k-4)\right) + 2 \cdot \left(2^{k-5} + 4 \cdot 2^{k-5} + 4 \cdot 2^{k-5}\right) \\ &\quad + 2 \cdot \left(\binom{k-4}{2} + 4 + 4 \cdot (k-4)\right) \\ &= 2^k + \frac{5}{2}k^2 - \frac{21}{2}k + 14. \end{aligned}$$

We now proceed to the proof of (3). Let $G \in \Gamma_k$ be such that $f(G) = M(k)$. By Lemma 1, $\Delta(G) \leqslant 3$.

Let $r = \lfloor \log_2(\sqrt{k}) + 1 \rfloor$ and $l = \lfloor k/2r \rfloor + 1$. By Lemma 3, there are $l$ disjoint sets $S_1, S_2, \ldots, S_l \subseteq E(G)$ each containing $r$ chords such that for $1 \leqslant i \leqslant l$ the chords in $S_i$ are either all pairwise skew or all pairwise parallel. Clearly, we have

$$\begin{aligned} f(G) &= \sum_{S \subseteq E(H) \setminus E(C_H)} g_G(S) \\ &\leqslant \left( \sum_{S \subseteq E(H) \setminus E(C_H)} 2 \right) - (2 - g_G(\emptyset)) - \sum_{i=1}^{l} \sum_{\emptyset \neq S \subseteq S_i} (2 - g_G(S)) \\ &= 2^{k+1} - 1 - \sum_{i=1}^{l} \left( 2^{r+1} - 2 - \sum_{\emptyset \neq S \subseteq S_i} g_G(S) \right). \end{aligned}$$

**Claim 3.** $2^{r+1} - 2 - \sum_{\emptyset \neq S \subseteq S_i} g_G(S) \geqslant 2^r - r(r-1) - 2$ *for* $1 \leqslant i \leqslant l$.

**Proof of Claim 3.** If the chords in $S_i$ are pairwise skew, then Lemma 2 (i) implies

$$2^{r+1} - 2 - \sum_{\emptyset \neq S \subseteq S_i} g_G(S) = 2^{r+1} - 2 - 2^r - 2\binom{r}{2} = 2^r - r(r-1) - 2.$$

If the chords in $S_i$ are pairwise parallel, then Lemma 2(ii) implies

$$2^{r+1} - 2 - \sum_{\emptyset \neq S \subseteq S_i} g_G(S) = 2^{r+1} - 2 - 2r - \binom{r}{2} \geqslant 2^r - r(r-1) - 2$$

and the proof of the claim is complete. $\square$

Altogether, we obtain

$$\begin{aligned} f(G) &\leqslant 2^{k+1} - 1 - l\left(2^r - r(r-1) - 2\right) \\ &\leqslant 2^{k+1} - 1 - \left( \left\lfloor \frac{k}{2r} \right\rfloor + 1 \right) \left(2^r - r(r-1) - 2\right) \\ &\leqslant 2^{k+1} - 1 - \frac{k}{2\left(\log_2\left(\sqrt{k}\right) + 1\right)} \left( \sqrt{k} - \left( \log_2\left(\sqrt{k}\right) + 1 \right) \log_2\left(\sqrt{k}\right) - 2 \right) \\ &= 2^{k+1} - 1 - k \left( \frac{\sqrt{k} - 2}{\log_2(k) + 2} - \frac{1}{4} \log_2(k) \right) \end{aligned}$$

and the proof is complete. $\square$

Note that the upper bound in (3) is better than Shi's upper bound in (1) only for large enough $k$.

## 3. Results on $M(k)$ for $5 \leqslant k \leqslant 10$

In this section we report computational results regarding $M(k)$ for $5 \leqslant k \leqslant 10$. In order to determine $M(k)$ and the structure of the extremal graphs for fixed $k$, Lemma 1 (iv)

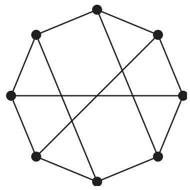

Fig. 1. Extremal graph for $k = 4$ with 29 cycles.

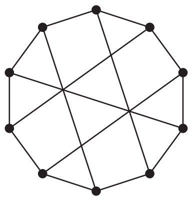

Fig. 2. Extremal graph for $k = 5$ with 56 cycles.

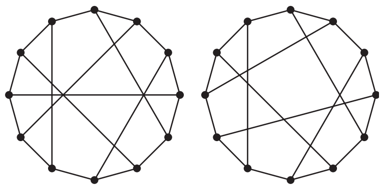

Fig. 3. Extremal graphs for $k = 6$ with 109 cycles.

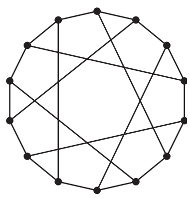

Fig. 4. Extremal graph for $k = 7$ with 213 cycles.

implies that we can restrict our considerations to Hamiltonian, 3-regular graphs of order $2k$ having $3k$ edges. Using a computer program we generated all such graphs and calculated the number of their cycles. This yields $M(k)$ and also the collection of extremal graphs possibly containing several copies of each graph. Using a generic isomorphism test, we determine all isomorphism types of extremal graphs (Figs.1–7). In the following figures, we report $M(k)$ and give representatives of these isomorphism types.

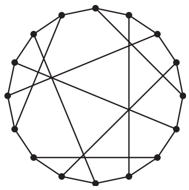

Fig. 5. Extremal graph for $k = 8$ with 401 cycles.

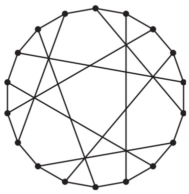

Fig. 6. Extremal graph for $k = 9$ with 783 cycles.

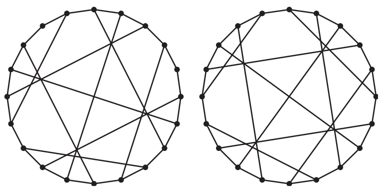

Fig. 7. Extremal graphs for $k = 10$ with 1484 cycles.

As we already mentioned in the introduction, we were not able to find some regularity in these configurations.

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