Data Structures and Graph Algorithms

Weighted Matchings

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K. Mehlhorn and G. Schäfer: Implementation of $O(nm\log n)$ Weighted Matchings in General Graphs: The Power of Data Structures, Workshop on Algorithm Engineering (WAE), LNCS 1982, 23–38, full version to appear in Journal of Experimental Algorithmics

K. Mehlhorn and G. Schäfer: A Heuristic for Dijkstra's Algorithm with Many Targets and its Use in Weighted Matching Algorithms, ESA 2001, LNCS 2161, 242–253,

Contents

1. the worst case running time of many graph algorithms can be considerably improved by clever data structures ($n =$ number of nodes, $m =$number of edges)

priority queues for shortest paths	$O(n^2)$	$\Longrightarrow$	$O(m + n\log n)$
dynamic trees for maximum flows	$O(n^2\sqrt{m})$	$\Longrightarrow$	$O(nm)$
mergeable p-queues for weighted matchings	$O(n^3)$	$\Longrightarrow$	$O(nm\log n)$

2. do these asymptotic improvements lead to improved “actual” running times ?
   - priority queues for Dijkstra’s shortest path algorithm
   - dynamic trees for maximum flow algorithms
   - mergeable priority queues for general weighted matchings ???

3. can we explain our experimental findings ???

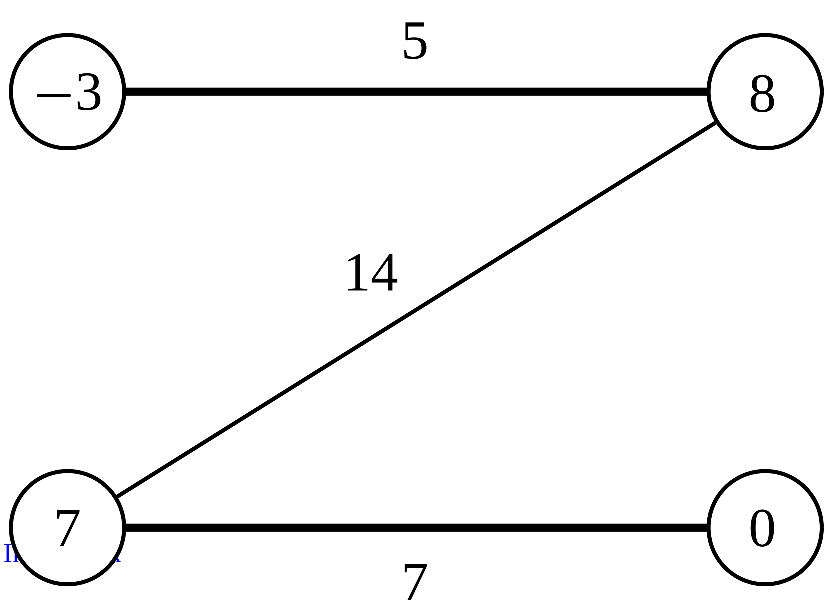
4. today’s talk is an engineering talk and not a theory talk

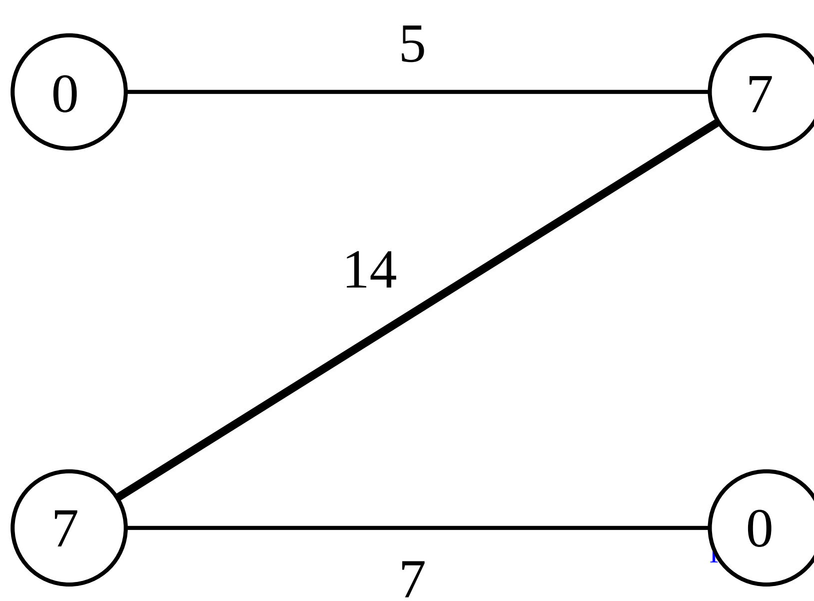
Worst Case Analysis of Graph Algorithms vs Actual Running Times

- view execution as a sequence of basic operations, for example
  - scan all edges incident to a node
  - find a node with with minimal priority
- derive an upper bound a_i on the number of operations of type i
- argue how operations of type i can be implemented and derive a time bound T_i
- state $\sum_i a_i T_i$ as an upper bound on the running time
- a_i and T_i are stated as functions of n and m (number of nodes and edges, resp.) and we are interested in the asymptotics
- theoretical bottlenecks = the i's such that $a_i T_i$ determines the asymptotics
- a_i and T_i are upper bounds
- t_i = typical number of operations of type i
- actual bottleneck = the i's such that $t_i T_i$ determines the running time

The Weighted Matching Problem

- $G = (V, E)$ graph, $W : E \mapsto \mathbb{R}$, edge costs
- matching M = set of edges no two of which share an endpoint
- M is perfect iff every node of G is matched
- weight of a matching $w(M) = \sum_{e \in M} w(e)$
- weighted matching problem
  - compute perfect M with maximum (minimum) weight
  - compute M with maximum (minimum) weight
  - problems are reducible to each other (but it is better to solve them directly)





Algorithms

- Edmonds (65) gave blossom-shrinking alg, running time $O(n^2 m)$
- Lawler (76) and Gabow (74) improved time to $O(n^3)$
- Galil, Micali, and Gabow (86) improved further to $O(nm \log n)$
- Gabow (90) improved further to $O(nm + n^2 \log n)$.
- underlying strategy is the same, algs use different data structures
  - Edmonds, Lawler, Gabow only require arrays and lists
  - Galil, Micali, and Gabow require mergeable and splitable priority queues
  - Gabow (90) requires even more sophisticated data structures
- **Question:** Is is worth using sophisticated data structures?

Implementations

- Applegate/Cook (93) and Cook/Rohe (97): Blossom IV
  - implement $O(n^3)$ algorithm
  - refined by several powerful heuristics: jump start, multiple trees, variable δ, pricing for complete geometric instances
  - argue convincingly that their implementation is best
- Mehlhorn/Schäfer (00 + 01)
  - implement variant of $O(nm \log n)$ algorithm
  - use mergeable and splitable priority queue
  - refined by some heuristics: multiple trees and improved jump start
  - weighted matchings or weighted perfect matchings
  - is usually faster than Blossom IV (except for dense geometric instances)
    - data structures make the worst case and the common case faster
  - available as part of LEDA and built on top of LEDA

Some Timings

Delaunay Graphs		
n	B4	MS
10000	2.69	4.88
40000	17.60	21.68
160000	138.29	98.13

Sweep Triangulations		
n	B4	MS
10000	9.89	5.16
40000	95.96	24.09
160000	2373.18	121.99

- Delaunay graphs and sweep triangulations are planar graphs
- Delaunay Graphs are simpler than general triangulations
- MS seems to have better asymptotics on these examples
  - quadrupling n increases running time by factor ≈ 5 for MS
  - and by factor 7 and more for B4.

More Timings

- random graphs, $n = 10^4$, $m = 4n$, random edge weights in $[1\,..\,b]$

b	B4	MS
1	3.99	0.85
100	3.10	2.58
10000	11.91	2.78

  running time of MS depends less on range of edge weights and is faster in the unweighted case

- random graphs , $n = 4 \cdot 10^4$, $m = 6n$

n	B4			MS		
	best	ave	worst	best	ave	worst
40000	49.02	55.15	60.74	9.93	10.30	11.09

  running time of MS shows less variance

More Timings

- a chain with $2n$ vertices and $2n-1$ edges. Edge weights are alternately 0 and 2 with the extreme edges having weight 0.

  heuristics construct matching consisting of the weight 2 edges,

  one augmentation is needed to change the matching into a max-weight perfect matching

n	B4	MS
10000	94.75	0.25
20000	466.86	0.64
40000	2151.33	2.08

  - running time of B4 grows quadratically
  - running time of MS slightly more than linearly

- a graph provided to us by Cook and Rohe: $n = 151780$ $m = 881317$

B4	MS
200810.35	5993.61

Optimality Condition: The Bipartite Case

Lemma 1 *A perfect matching M is optimal iff there are node potentials $(y_u)_{u \in V}$ with*

- $y_u + y_v \geq w_{uv}$ *for all* $uv \in E$
- $y_u + y_v = w_{uv}$ *for all* $uv \in M$

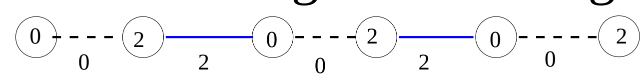
Let N be any other perfect matching. Then

$$w(N) = \sum_{e \in N} w_e \leq \sum_{u \in A} y_u + \sum_{v \in B} y_v = \sum_{e \in M} w_e = w(M)$$

reduced cost of edge uv $\qquad \pi_{uv} = y_u + y_v - w_{uv}$

an edge is called tight, if its reduced cost is equal to 0.

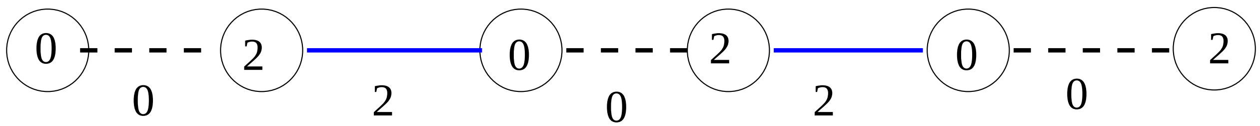
A High-Level View of Edmonds' Algorithm

- maintains a matching M and node potentials $(y_u)_{u \in V}$
  - reduced edge costs are non-negative: $\pi_{uv} = y_u + y_v - w_{uv} \geq 0$
  - edges in M are tight: $\pi_{uv} = 0$ for $uv \in M$
- operates in phases; in each phase $|M|$ is increased by one
- a phase consists of subphases
  - trees of tight edges rooted at free nodes are grown with the goal of finding an augmenting path connecting two free nodes 

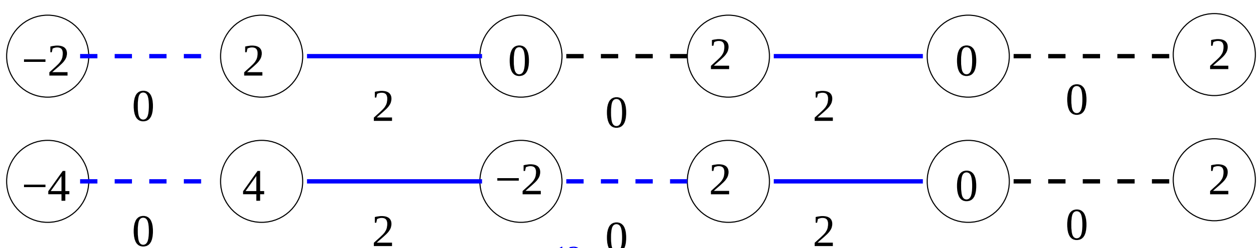
  - (subphase) when tree growing process stops, dual variables are changed to make more edges tight and hence to allow further growth
  - when augmenting path is found, $|M|$ is increased and phase ends.
- tree growing process may stop $\Omega(n)$ times and dual variable update may require to change $\Omega(n)$ potentials with primitive data structures: time $\Omega(n^2)$ per phase
- priority queues reduce cost of dual update to $O(\log n)$ time $O(m \log n)$ per phase

The Chain Example

- consider an odd length chain; edges in M are drawn solid, node potentials are shown
  reduced cost of edge $e = uv$ is $\pi_{uv} = y_u + y_v - w(e)$



- matching edges are tight
- we wish to construct an alternating path of tight edges starting at the left end point
  1. determine nodes reachable via tight edges (only left endpoint at the beginning)
  2. if free node (different from left endpoint) is reachable, stop and augment
  3. decrease potential of nodes at even distance by δ and increase potential of nodes at odd distance by δ, where $\delta = 2$
  4. goto step 1)

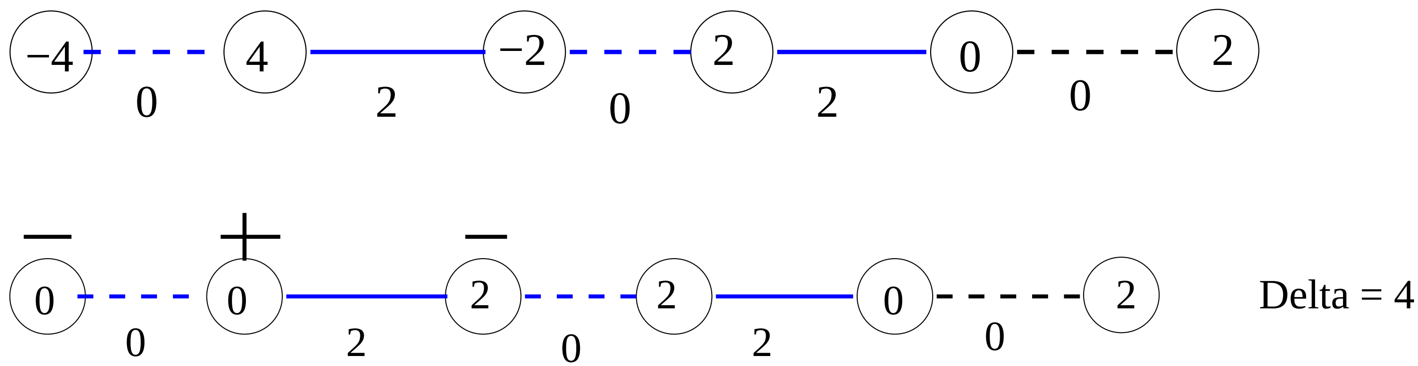


The Chain Example Made Fast

- reduced cost of edge $e = uv$ is $\pi_{uv} = y_u + y_v - w(e)$
- we wish to construct an alternating path of tight edges starting at the left end point
- keep a global offset Δ: for marked nodes v

true potential of v = stored potential of $v \pm \Delta$

$+\Delta$ for odd nodes, $-\Delta$ for even nodes, initially, $\Delta = 0$



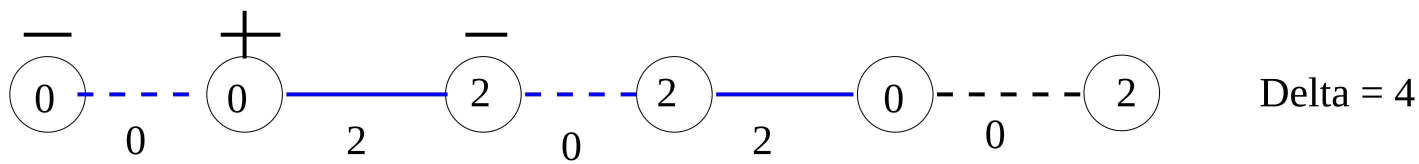
- Δ applies to marked nodes
- both figures define the same potential function

The Chain Example Made Fast II

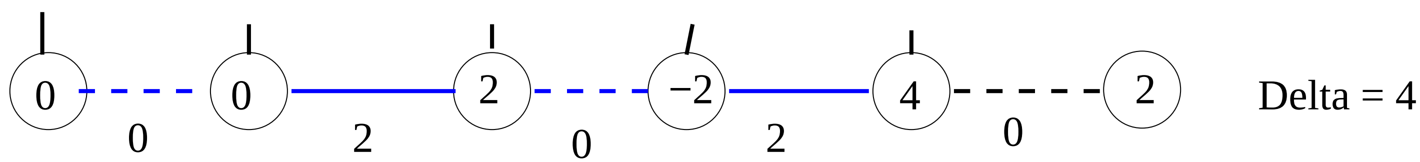
- keep a global offset Δ: for a node v reachable from left endpoint via tight edges

  true potential of v = stored potential of $v \pm \Delta$

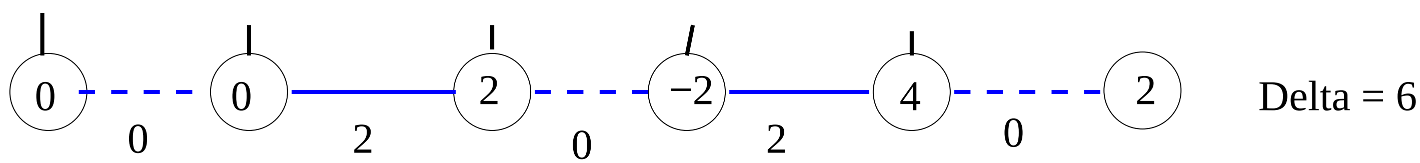
  $+\Delta$ for odd nodes, $-\Delta$ for even nodes, initially, $\Delta = 0$



- Δ applies to marked nodes
- new nodes are reachable via tight edges, we set their stored potential and mark them



- we change Δ so as to make one more edge tight: $\Delta = \Delta + \delta$, where $\delta = 2$



- potential update in time $O(1)$

Profiling Data

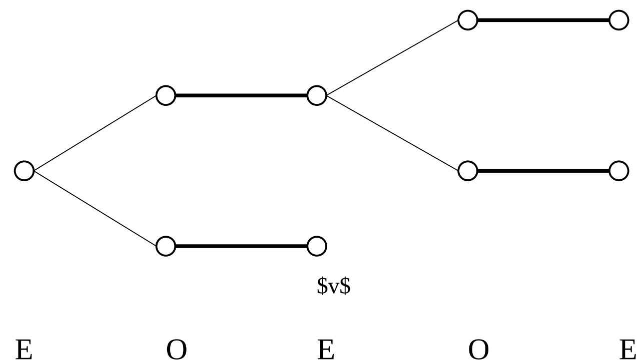
- B4 spends most of its time in potential updates and work triggered by it
- potential updates are not only theoretical bottleneck, they are the actual bottleneck for the n^3 algorithms
- offset trick allows us to change the potential of many nodes with a few instructions
- data structures keep the cost of other basic operations low
  - determining δ and determining new tight edges
- the following table gives numbers for sweep-triangulations, $n = 40000$

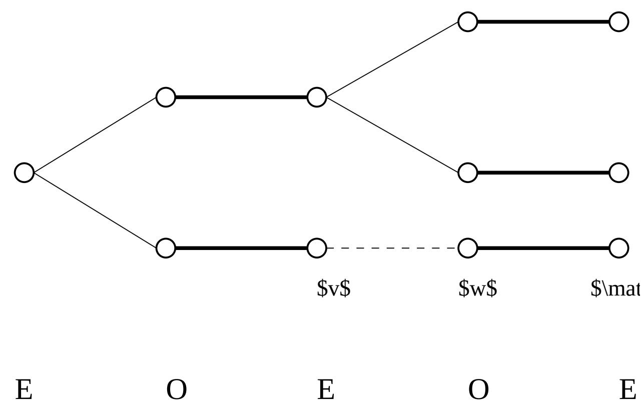
	dual adjustments	node updates	shrinks	expands	time
B4	12466	1179070	4913	433	23.41
MS	15812		7058	1766	21.98
B4	22488	11353087	10585	967	278.25
MS	24505		12140	2573	26.05

- we cannot use the variable δ-heuristic of BIV

More Details: Alternating Trees in Bipartite Case

- trees are rooted at free nodes and tree edges are tight
- vertices on even levels are reached via matching edges, vertices on odd levels are reached via non-matching edges
- action if there is tight edge (v, w) with v even
  - w is in no tree $\rightarrow$ grow tree, i.e., add w and its mate to the tree
  - w is even and in different tree: breakthrough
  - w is odd: do nothing





- if there is no such edge: potential change by δ, where

  $\delta =$ min reduced cost of any edge connecting a non-tree node with an even tree node

Priority Queues

maintain a set of priorities, under the following operations

- create an empty priority queue
- insert a priority
- delete a priority (given by a pointer to its position in the data structures)
- extract minimum priority
- decrease a priority (given by a pointer to its position in the data structures)

- there are priority queue implementations supporting all operations in logarithmic time
- even time $O(1)$ for *decrease_p*

Usage of Priority Queues: Bipartite Case

- $\delta =$ min reduced cost of any edge connecting a non-tree node with an even tree node
- for every non-tree node v keep

  $min\_cost(v)$ = minimum reduced cost of any edge connecting v to an even tree node
- keep all *min_cost*-values in a priority queue
- tree growing and dual updates
  - *extract_min* determines δ and the node to be added to the tree
  - delete new tree nodes from the priority queue
  - update priorities of remaining non-tree nodes by scanning the edges incident to the new even tree node u
  - for every such edge uv check $min\_cost(v)$ and decrase if necessary
  - after dual update, tree will grow by at least two nodes and hence there are at most $n/2$ dual updates per phase
  - cost of one dual update = 1 *extract_min* + $deg(u)$ *decrease_p*
  - total cost of dual updates per phase = $O(n \log n + m)$

An Optimization (ESA 01)

- grow a single tree
- "conceptually" combine all free non-tree nodes into a single node and use them to derive a threshold for pq-operations
- three additional lines of code
- considerably reduces number of queue operations
- about halves the running time

Lemma 2 (MS, ESA 01) *On random graphs with average outdegree c the fraction of saved queue operations is at least*

$$1 - \frac{2 + \ln c}{c} \ .$$

For example, for $c = 8$, at least 49% of the queue operations are saved, and
for $c = 16$, at least 70% are saved.

Lemma 3 (Optimality Conditions for the General Case)
A perfect matching M is optimal iff there are node potentials $(y_u)_{u \in V}$ and odd set potentials $(z_{\mathcal{B}})_{\mathcal{B} \in \mathcal{O}}$ with

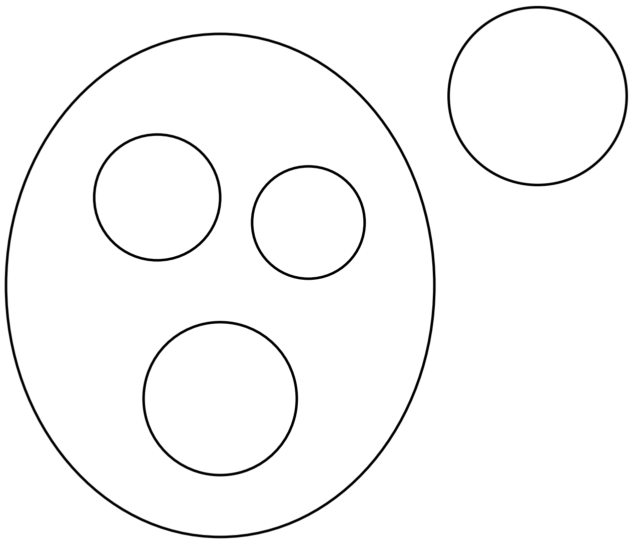
- $z_{\mathcal{B}} \geq 0$ *for all* $\mathcal{B} \in \mathcal{O}$,
- $\pi_{uv} = y_u + y_v - w_{uv} + \sum_{uv \subseteq \mathcal{B}} z_{\mathcal{B}} \geq 0$ *for all* $uv \in E$
- $\pi_{uv} = 0$ *for all* $uv \in M$ *(matching edges are tight)*
- *sets* $\mathcal{B}$ *with* $z_{\mathcal{B}} > 0$ *form a nested family and are full, i.e., exactly one node has mate outside* $\mathcal{B}$ *and there are* $\lfloor |\mathcal{B}|/2 \rfloor$ *matching edges inside.*

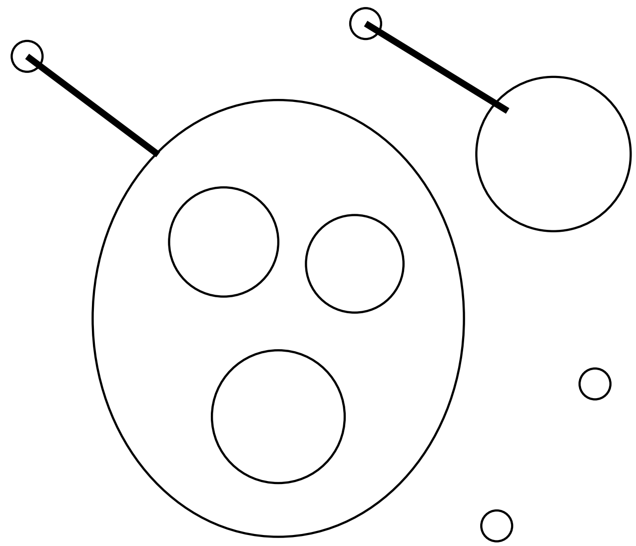
Let N be any other perfect matching. Then

$$\begin{aligned} w(M) = \sum_{e \in M} w_e &= \sum_{u \in V} y_u + \sum_{uv \in M} \sum_{\mathcal{B}; uv \in \mathcal{B}} z_{\mathcal{B}} \\ &= \sum_{u \in V} y_u + \sum_{\mathcal{B}; z_{\mathcal{B}} > 0} z_{\mathcal{B}} \lfloor |\mathcal{B}|/2 \rfloor \\ &\geq \sum_{u \in V} y_u + \sum_{uv \in N} \sum_{\mathcal{B}; uv \in \mathcal{B}} z_{\mathcal{B}} \geq \sum_{e \in N} w_e = w(N) \end{aligned}$$

Edmonds' Algorithm: More Details

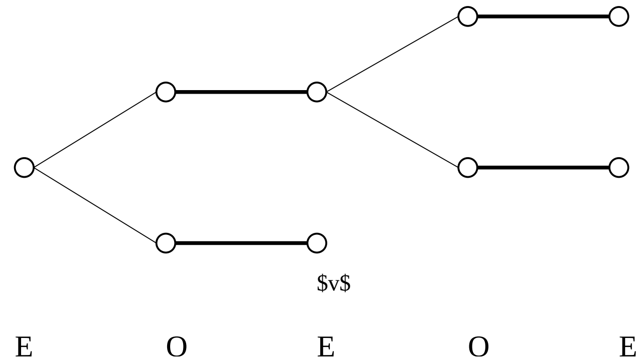
- alg maintains a matching M and dual variables y_v, $z_{\mathcal{B}}$
- all edges have non-negative reduced cost
- matching edges are tight
- init: $M = \emptyset$, $z_{\mathcal{B}} = 0$ for all $\mathcal{B}$, $y_v = \max_{e \in E} w_e$
- sets $\mathcal{B}$ with $z_{\mathcal{B}} > 0$ are called blossoms and form a nested family of full sets
- surface blossom = maximal blossom
- vertices of current graph =
  - nodes of original graph outside blossoms
  - surface blossoms
- surface blossoms are matched in current graph
- free vertices are original nodes
- alg grows alternating trees rooted at free vertices

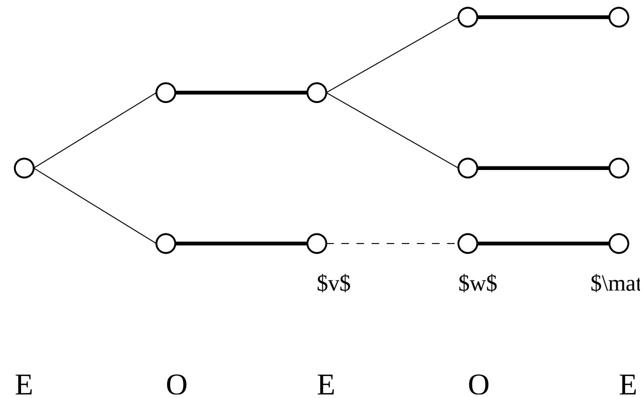


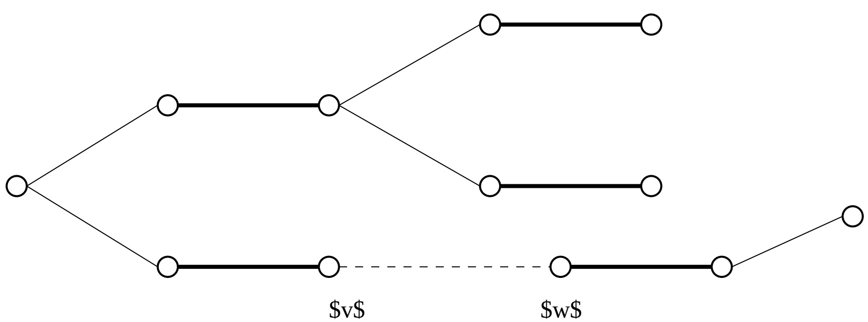


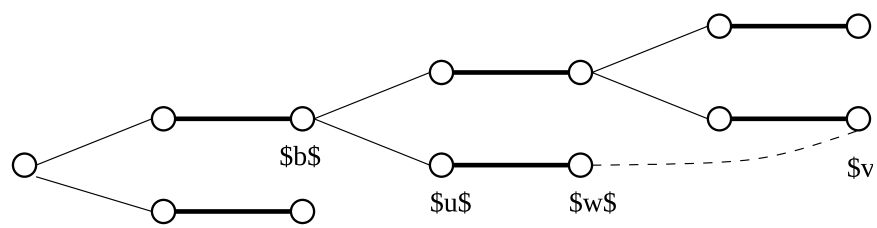
Alternating Trees

- all edges in the tree are tight
- vertices on even levels are reached via matching edges, vertices on odd levels are reached via non-matching edges
- actions if there is tight edge (v,w) with v even
  - w is in no tree: add w and its mate
    w and/or its mate may be surface blossoms
  - w is even and in different tree: breakthrough
  - w is even and in same tree: new even blossom
  - w is odd: do nothing







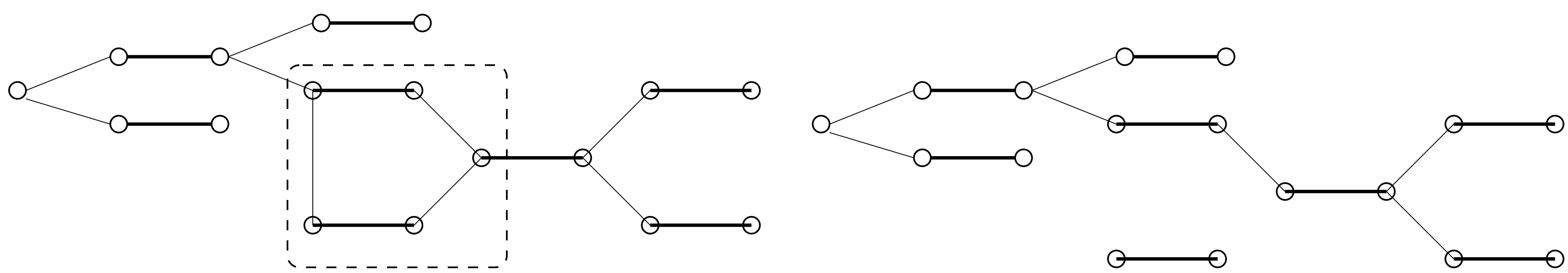


Dual Updates I

- when tree growing process stops without a breakthrough, update the duals
- goal is to make edges tight that connect non-tree nodes with even tree nodes
- $y_v = y_v - \delta$ for even nodes, $y_v = y_v + \delta$ for odd nodes
- $z_{\mathcal{B}} = z_{\mathcal{B}} + 2\delta$ for even surface blossoms, $z_{\mathcal{B}} = z_{\mathcal{B}} - 2\delta$ for odd surface blossoms
- does not change the reduced cost of any edge
  - inside a blossom or in alternating trees
- decreases the reduced cost of edges between non-tree nodes and even tree nodes, and between even tree nodes and the potential of odd surface blossoms
- constraints on δ
  - $\delta \le \delta_1 = \min\{\pi_{uv}\ ;\ u \text{ even}, v \text{ non-tree}\}$
  - $\delta \le \delta_2 = \min\{\pi_{uv}/2\ ;\ u \text{ and } v \text{ even}\}$
  - $\delta \le \delta_3 = \min\{z_{\mathcal{B}}/2\ ;\ \mathcal{B} \text{ is an odd surface blossom}\}$
  - choose $\delta = \min(\delta_1, \delta_2, \delta_3)$.

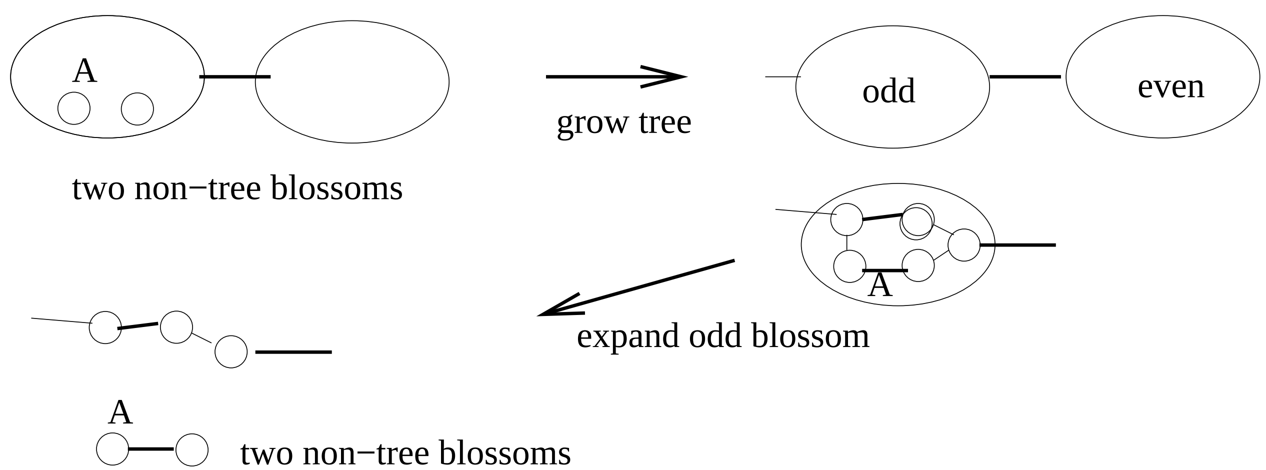
Dual Updates II

- $y_v = y_v - \delta$ for even nodes, $y_v = y_v + \delta$ for odd nodes
- $z_{\mathcal{B}} = z_{\mathcal{B}} + 2\delta$ for even surface blossoms, $z_{\mathcal{B}} = z_{\mathcal{B}} - 2\delta$ for odd surface blossoms
- constraints on δ
  - $\delta \leq \delta_1 = \min\{\pi_{uv}\ ; u$ even, v non-tree$\}$
  - $\delta \leq \delta_2 = \min\{\pi_{uv}/2\ ; u$ and v even$\}$
  - $\delta \leq \delta_3 = \min\{z_{\mathcal{B}}/2\ ; \mathcal{B}$ is an odd surface blossom$\}$
  - choose $\delta = \min(\delta_1, \delta_2, \delta_3)$.
  - if $\delta = \delta_1$, grow tree
  - if $\delta = \delta_2$, breakthrough or new even blossom
  - if $\delta = \delta_3$, expand odd surface blossom



Mergeable and Splitable Priority Queues

- priority queues are needed to keep track of various δ's
- mergeable and splitable priority queues
  - priorities are associated with elements of sequences
  - have a priority queue for each sequence
  - sequences can be concatenated and split
- edges from A to even tree nodes must be considered in first and last figure, but not in the middle figure



Summary and Open Problems

- have carefully implemented variant of $O(nm \log n)$ algorithm
- a significant practical contribution, a small theoretical advance
  - first implementation of the algorithm
  - simplified treatment of reduced costs
  - optimization for bipartite graphs, analysis thereof
- new impl is superior to previous ones
- impl is based on LEDA and would have been impossible without

- how about Gabow's $O(nm + n^2 \log n)$ alg?
- design families of problem instances which force algs into their worst case
- transfer optimization from bipartite graphs to general graphs
- explain asymptotic behavior on Delaunay graphs, sparse random graph