Euler and Hamilton Paths

The town of Königsberg, Prussia (now know as Kaliningrad and part of the Russian republic), was divided into four section by branches of the Pregel River. These four sections

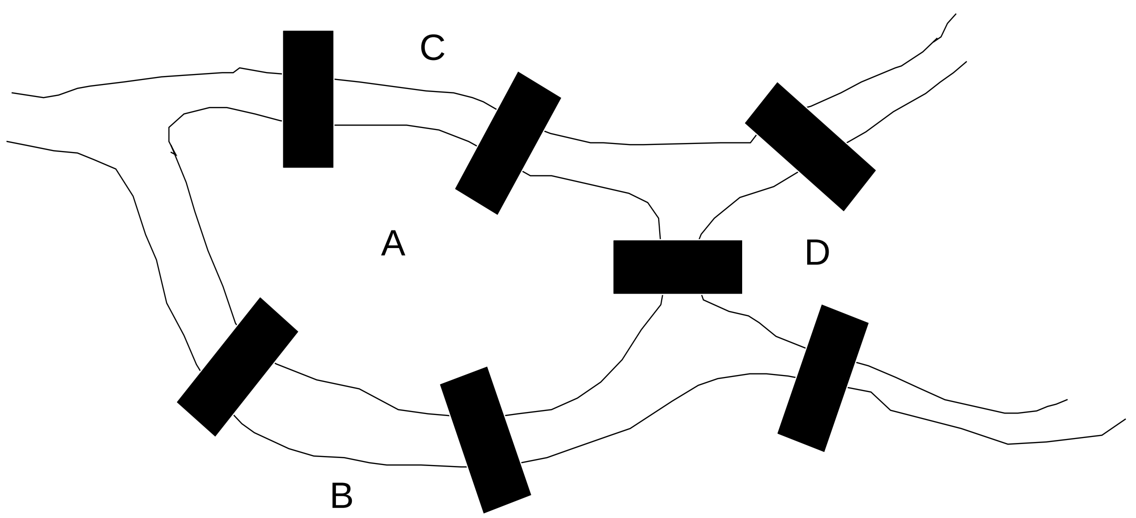


Figure: Seven Bridges of Koinigsberg

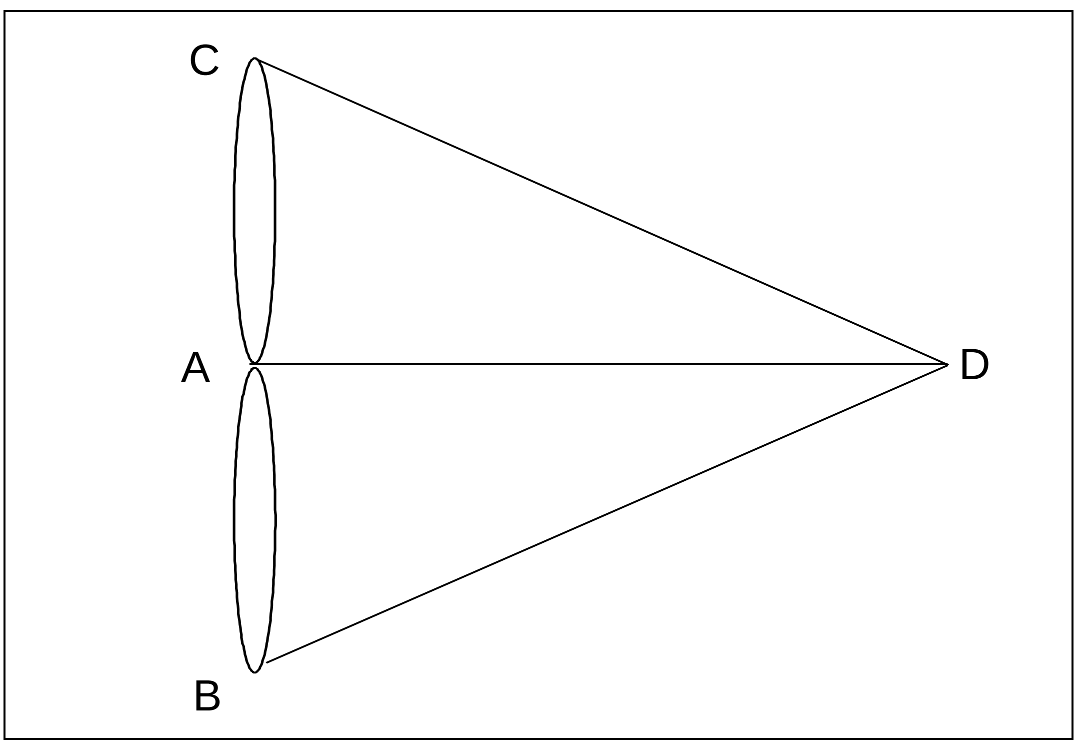
included the two regions on the banks of the Pregel, Kneiphof Island and the region between the two branches of the Pregel. In the eighteenth century, seven bridges connected these regions.

The following diagram shows the position of the bridges on the Pregel in the town of Königsberg.

As for an European country, the town folks used to walk round the town on the Sundays, and every time they wondered if there was any means to go round the city without crossing these bridges twice, and return to the starting point.

Swiss mathematician Leonhard Euler solved this problem and published that in 1736. In his solution, he used graphs for the first time to solve this problem. His key points in the solutions were taking the regions as vertices, and the bridges as the edges between the vertices – thus making a multigraph.

His multigraph solution is in the following figure:



This diagram rephrased or showed the problem of travelling across every bridge without crossing any bridge more than once. Now, we may ask if we can draw a simple circuit from this multigraph that will contain all the bridges in the original diagram? That means, can we draw a simple graph that contains every edge of the multigraph.

Euler Circuit: An *Euler Circuit* in a graph G is a simple circuit containing every edge of G.

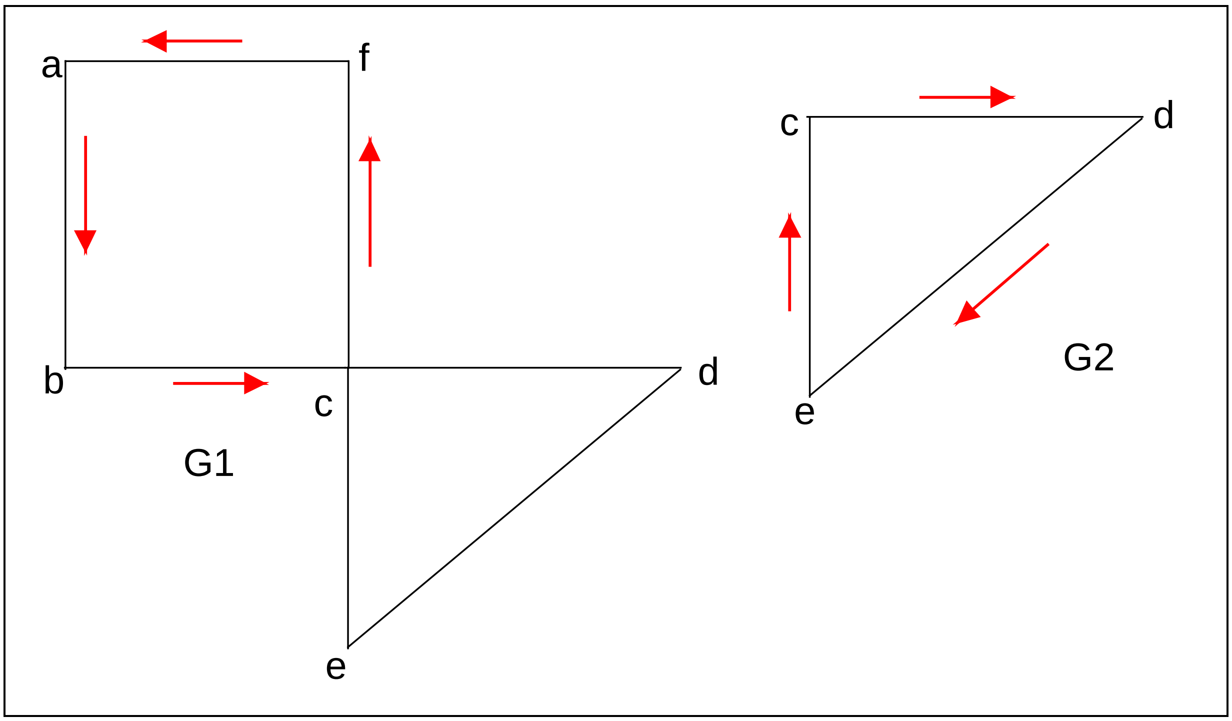
Euler Path: An *Euler Path* in G is a simple path containing every edge of G.

Necessary and Sufficient Conditions for Euler Circuits and Paths

What is the condition if a connected multigraph has an Euler circuit? This is done if we can show that every vertex must have *even degree*.

Let us suppose that one Euler circuit begins with a vertex a, and continues with an edge incident to a, say $\{a, b\}$. The edge contributes 1 to *deg*(a). Each time the circuit passes through a vertex it contributes 2 to the vertex's degree. This is because the circuit enters from an edge incident with this vertex and leaves via another such edge. Finally the circuit terminates at the point it originated: say at vertex *a*, contributing 1 to *deg*(a).

Therefore, *deg*(a) must be even, because the circuit contributes 1 when it begins, and 1 when it ends, and 2



every time it passes through *a*. A vertex other than *a* has even degree because the circuit contributes 2 to its degree each time it passes through the vertex.

We conclude that if a connected graph has an Euler circuit, then every vertex must have even degree. Now we shall try to investigate if this necessary conditions is also sufficient for making an Euler circuit.

Suppose that G_1 is a connected graph. All its vertices have even degree values.

In the above graph G_1, the direction of the connected edges are: $\{a, b\}$, $\{b, c\}$, $\{c, f\}$, $\{f, a\}$. That is, the edges of the circuit begins from a, then passes through the vertices b, c, f and ultimately ends in a. If we study this graph carefully, we see that one edge enters to a vertex and then use another edge to come out from that vertex. That is why, for any vertex of the directed path of G_1 graph, there is an edge that is incident on that vertex (enters) and incident from that vertex (comes out).

If we consider vertex f, we see $\{c, f\}$ and $\{f, a\}$ are two edges that are incident to this vertex. Therefore, the degree of this vertex is 2, *i.e.*, $deg(\mathrm{f}) = 2$.

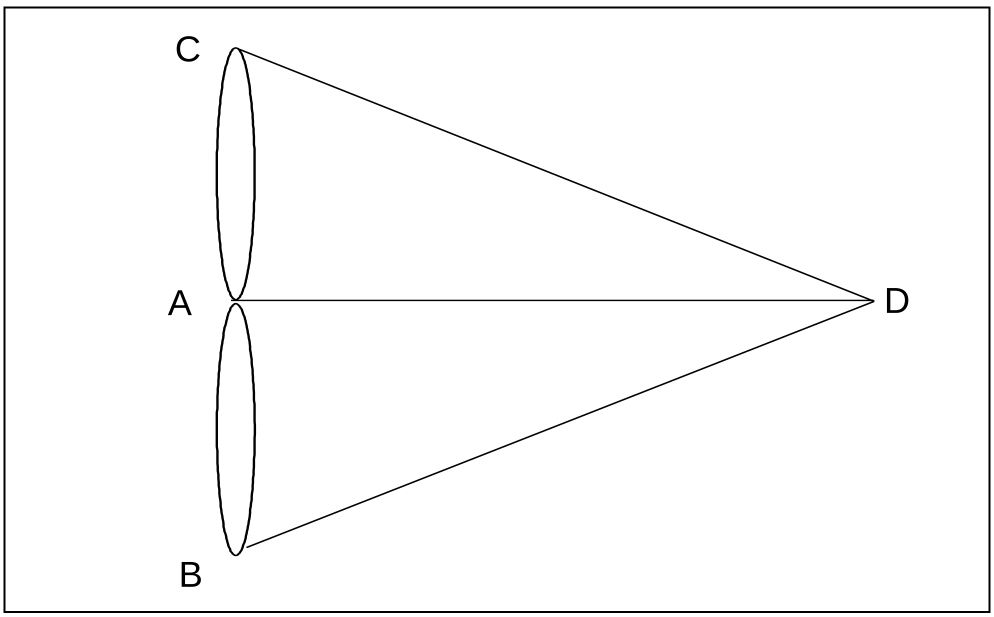
In the graph G_1, we did not show any direction for the other smaller connected paths for the vertices c, d, e. Let us suppose that these vertices $\{c, d, e\}$ makes up another subgraph if we delete vertices $\{a, b, f\}$. Let us name this subgraph as G_2.

Now, let us suppose a circuit in the graph G_2 as we had drawn in G_1. Let the circuit begins from vertex c, then goes to d, then to e, and finally ends in c. The circuit is: c, d, e, c. And they make a path in the graph G_2.

If we consider the graph G_1, the circuit would be: a, b, c, d, e, f, a.

Theorem: A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.

Königsberg Bridge Problem



Now let us try to solve the Königsberg bridge problem. Let us see the Euler's graph of the Königsberg bridges here again:

This graph has four vertices of odd degree: deg (a) = 5, deg (b) = 3, deg (c) = 3 and deg (d) = 3. So it does not have an Euler circuit. There is no way to start from one point, cross each bridge once, and then come back to the starting point.

Fleury's Theorem: If the originating vertex of a multigraph is arbitrarily chosen, it forms a circuit by choosing edges successively. Once an edge is chosen, that edge is removed. Edges are chosen successively so that each edge begins where the last edge ends, and so that this edge is not a cut edge unless there is no alternative.

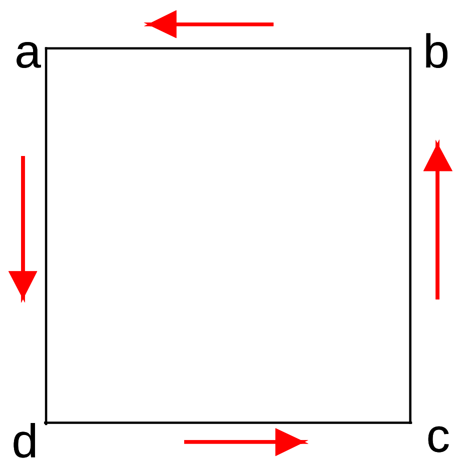
The theorem in logical steps is given below:

Step 1: Choose a starting vertex, say *u*.
Step 2: Traverse any available edge, choosing an edge that will disconnect the remaining graph only if there is no alternative.
Step 3: After traversing each edge, remove it (together with any vertices of degree 0 which result)
Step 4: If no edge remains, stop. Otherwise, choose another available edge and go back to step 2.

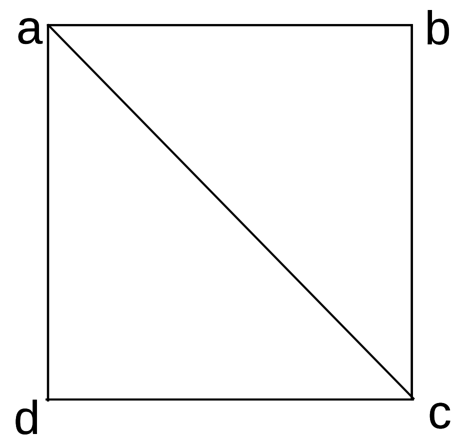
Theorem 2: A connected multigraph has an Euler path but not an Euler circuit if and only if has exactly two vertices of odd degree.

Let us suppose the following graphs G and H:

The graph *G* is an *Euler Circuit*: all the vertices are of degree 2.



G



H

The graph *H*, according to the theorem 2, has *Euler path* but has no *Euler circuit*, as two of its vertices are of degree 3. Those tow vertices are *a* and *c*.

Hamiltonian Paths and Circuits

So far we had developed necessary and sufficient conditions for the existence of paths and circuits that contain every edge of a multigraph exactly once. Here, with the help of Hamiltonian Paths and Circuits, we shall try to see if there exists any simple paths and circuits containing every vertex of the graph exactly once.

This terminology comes from a puzzle invented in 1857 by an Irish mathematician Sir William Rowan Hamilton. He was devoted to non-commutative algebra, and worked a lot in this area. Though, he made important contributions to the optics, abstract algebra and dynamics. He invented “Icosian Game” based on his work in non-commutative algebra. The puzzle in the example 5 is the representation of that game.

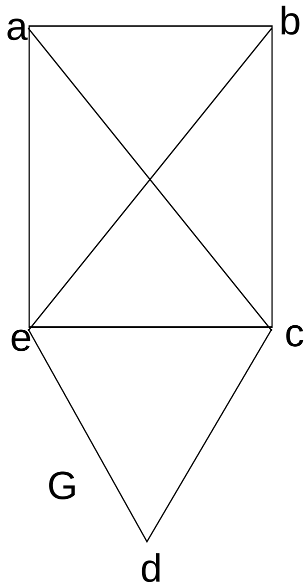
Before moving to examples, let us see the definition of Hamiltonian Path and Hamiltonian Circuit.

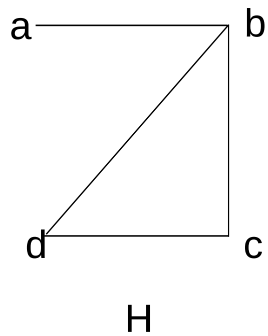
Hamiltonian Path: A path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is called ***Hamiltonian Path*** if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i \leq j \leq n$.

Hamiltonian Circuit: A circuit $x_0, x_1, \ldots, x_{n-1}, x_n$ (with $n>1$) in a graph $G = (V, E)$ is called a ***Hamiltonian circuit*** if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamiltonian Path.

Q_n is a ***Hamiltonian circuit***.

Let us suppose a simple graph G as depicted in the following figures:





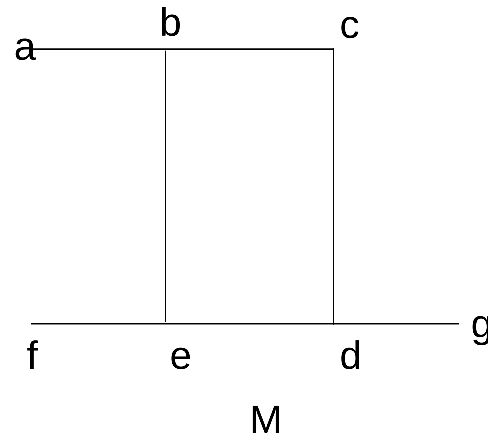


Figure *G*: The ***Hamiltonian circuit*** is present in this graph. The circuit is a, b, c, d, e, a.

Figure H: No Hamiltonian circuit is present: for a Hamiltonian circuit be present here, that must contain the edge $\{a, b\}$ twice.
But there is a ***Hamiltonian Path***: a, b, c, d.

Figure M: No ***Hamiltonian Path*** or ***Hamiltonian Circuit*** is present here. At least one vertex is left out of Hamiltonian Path, and circuit is not possible.

<u>Some necessary and sufficient conditions for Hamiltonian Circuits and Paths</u>

There are no known simple necessary and sufficient theorems for the existence of Hamiltonian circuits.

Still there are some properties can be devised that may provide some give necessary conditions for Hamiltonian circuits. One of such property says that: ***A graph with a vertex of degree 1 cannot have a* Hamiltonian Circuit**. This is only because a Hamiltonian circuit, each vertex is incident with two edges in the circuit.

Another property says that ***if a vertex in the graph has degree 2, then both edges that are incident with this vertex must be part of any* Hamiltonian Circuit**.

Also note that, ***when a Hamiltonian circuit is being constructed and this circuit has passed through another vertex, then all remaining edged incident with this vertex, other than the two used in the circuit, can be removed from consideration***.

Furthermore ***a Hamiltonian circuit cannot contain a smaller circuit within it***.

Sufficient conditions for existence of* Hamiltonian circuit*:

If G is a connected simple graph with n vertices where $n \geq 3$, then G has a ***Hamilton circuit*** if the degree of each vertex is at least $n/2$.